Section 16.3
Conservative Vector Fields

FTC for Conservative Vector Fields

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1 FTC for Conservative Vector Fields

Conservative Vector Fields

Recall from §16.1 that a vector field $\vec{F}$ is **conservative** if it has a **scalar potential**, i.e., a function f such that $\nabla f = \vec{F}$.

- If $\vec{F}$ is conservative on an open connected domain, then any two scalar potentials of $\vec{F}$ differ by a constant.
- Potentials can be calculated by the "antidifferentiate and match up the pieces" method.
- If $\vec{F}$ is conservative, then $\text{curl}(\vec{F}) = \vec{0}$.

Fundamental Theorem for Conservative Vector Fields

Assume that $\vec{F} = \nabla f$ on an open connected domain $\mathcal{D}$.

If $\vec{r}$ is a path along a curve $\mathcal{C}$ from P to Q in $\mathcal{D}$, then

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = f(Q) - f(P).$$

- If $\vec{r}$ is a path on the x-axis, then this result reduces to FTC.

Fundamental Theorem for Conservative Vector Fields

Assume that $\vec{\mathrm{F}} = \nabla f$ on an open connected domain $\mathcal{D}$. If $\vec{\mathrm{r}}$ is a path along a curve $\mathcal{C}$ from P to Q in $\mathcal{D}$, then

$$\int_{\mathcal{C}} \vec{\mathrm{F}} \cdot d\vec{\mathrm{r}} = f(Q) - f(P)$$

Proof: Assume $\vec{\mathrm{r}}(a) = P$ and $\vec{\mathrm{r}}(b) = Q$.

$$\int_{\mathcal{C}} \vec{\mathrm{F}} \cdot d\vec{\mathrm{r}} = \int \nabla f \cdot d\vec{\mathrm{r}} = \int_a^b \nabla f(\vec{\mathrm{r}}(t)) \cdot \vec{\mathrm{r}}'(t)\, dt$$

$$= \int_a^b \frac{d}{dt}\Big(f(\vec{\mathrm{r}}(t))\Big)\, dt \qquad \text{(by Chain Rule)}$$

$$= f(\vec{\mathrm{r}}(t))\Big|_{t=a}^{t=b} \qquad \text{(by FTC)}$$

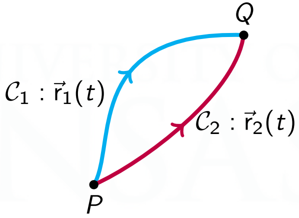
$$= f(\vec{\mathrm{r}}(b)) - f(\vec{\mathrm{r}}(a)) = f(Q) - f(P).$$

Note: The proof works for any open domain; we assume a connected domain for conservative fields to keep the potential functions the same up to a constant.

2 Properties of Conservative Vector Fields

Properties of Conservative Vector Fields

Consequence 1: If $\mathcal{C}$ is a **closed** curve (i.e., $P = Q$), then $\int_{\mathcal{C}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} = 0$.



Consequence 2: If $\mathcal{C}_1$ and $\mathcal{C}_2$ are paths in $\mathcal{D}$ from P to Q, then

$$\int_{\mathcal{C}_1} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} = \int_{\mathcal{C}_2} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}}$$

That is, conservative vector fields are **path-independent:** line integrals **depend only on the endpoints** of the path of integration. (This is not in general true for non-conservative fields!)

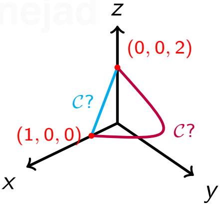
Path-Independence of Line Integrals

Example 1: Suppose $\vec{\mathsf{F}} = \nabla f$ where $f(x,y,z) = \dfrac{-1}{x^2+y^2+z^2}$. Find the work done by $\vec{\mathsf{F}}$ in moving an object along a smooth curve $\mathcal{C}$ from $(1,0,0)$ to $(0,0,2)$ without passing through the origin.

Solution: Since $\vec{\mathsf{F}}$ is conservative on $\mathbb{R}^3 - \{(0,0,0)\}$, the Fundamental Theorem for Conservative Vector Fields says that

$$\int_{\mathcal{C}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} \;=\; f(0,0,2) - f(1,0,0) \;=\; \frac{3}{4}.$$

You don't need to know the path $\mathcal{C}$ because the potential function is given!



Path-Independence of Line Integrals

- Conservative vector fields have path-independent line integrals.
- Conversely, suppose $\vec{F}$ has path-independent line integrals in an open connected domain $\mathcal{D}$. Pick a starting point $P \in \mathcal{D}$. For every $Q \in \mathcal{D}$, let $\mathcal{C}_Q$ be any path in $\mathcal{D}$ from P to Q.

  Then the function $f : \mathcal{D} \to \mathbb{R}$ defined by

  $$f(Q) = \int_{\mathcal{C}_Q} \vec{F} \cdot d\vec{r}$$

  turns out to be a scalar potential function for $\vec{F}$.

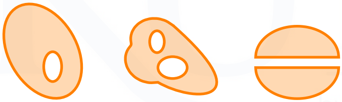
Theorem

A vector field $\vec{F}$ on an open path-connected domain $\mathcal{D}$ is path-independent if and only if it is conservative.

3 Simply-Connected Domains, the Theorem

Conservative Fields and Simply-Connected Domains

- We know that conservative vector fields are irrotational.
  I.e., if $\vec{F}$ is conservative then $\text{curl}(\vec{F}) = \vec{0}$.
- **Question:** Is every irrotational vector field conservative?



Examples of regions in $\mathbb{R}^2$ that are not simply-connected.

A path-connected domain $\mathcal{D}$ in $\mathbb{R}^2$ is **simply connected** if it has no holes. (More precisely, every loop in $\mathcal{D}$ can be contracted to a point while staying in $\mathcal{D}$.)

Theorem

Let $\vec{F}$ be a vector field on a **simply connected** domain $\mathcal{D}$.
If $\text{curl}(\vec{F}) = \vec{0}$ in $\mathcal{D}$, then $\vec{F}$ is conservative.

Conservative Fields and Simply-Connected Domains

Example 2: Is $\vec{F}(x,y) = \left\langle 3 + 2xy, x^2 - 3y^2 \right\rangle$ conservative? Evaluate $\displaystyle\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ when $\mathcal{C}$ is a path from $(1,1)$ to $(2,1)$.

<u>Solution:</u> The domain of $\vec{F}$ is $\mathbb{R}^2$, which is simply connected.

$$\text{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3 + 2xy & x^2 - 3y^2 & 0 \end{vmatrix} = (2x - 2x)\vec{k} = \vec{0}$$

So $\vec{F}$ is conservative by the previous theorem.

Reminder: find a potential f by antidifferentiating and matching pieces:

$$\begin{aligned} f(x,y) &= \int \vec{F}_1 \, dx = \int 3 + 2xy \, dx = 3x + x^2y + a(y) \\ &= \int \vec{F}_2 \, dy = \int x^2 - 3y^2 \, dy = x^2y - y^3 + b(x) \end{aligned}$$

$$f(x,y) = \underbrace{3x + x^2y - y^3}_{\text{A potential}} + C$$

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = f(2,1) - f(1,1) = (6 + 4 - 1 + C) - (3 + 1 - 1 + C) = 6$$

Conservative Fields and Simply-Connected Domains

Example 3: Is $\vec{F}(x,y) = \left\langle \dfrac{-y}{x^2+y^2},\ \dfrac{x}{x^2+y^2} \right\rangle$ conservative?

Answer: **No,** because it is not path-independent. If $\mathcal{C}$ is the unit circle, with standard parametrization $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \le t \le 2\pi$, then

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \overbrace{\langle -\sin(t), \cos(t) \rangle}^{\vec{F}(\vec{r}(t))} \cdot \overbrace{\langle -\sin(t), \cos(t) \rangle}^{\vec{r}\,'(t)} dt = \int_0^{2\pi} dt = 2\pi.$$

On the other hand,

$$\frac{dF_1}{dy} = \frac{dF_2}{dx} = \frac{y^2 - x^2}{(x^2+y^2)^2} \quad \therefore \quad \operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = \vec{0}.$$

What is going on here? The domain of $\vec{F}$ is $\mathbb{R}^2 - \{(0,0)\}$, which is not simply connected!

4 Applications in Physics

Conservative Vector Fields in Physics (Optional)

Let $\vec{F}$ be a continuous force field which moves an object of mass m along a path $\mathcal{C}$ parametrized by $\vec{r}$ from $A = \vec{r}(a)$ to $B = \vec{r}(b)$.

According to Newton's Second Law of Motion, $\vec{F}(\vec{r}(t)) = m\,\vec{r}''(t)$.

The work done by $\vec{F}$ on the object is

$$\begin{aligned}\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} &= \int_a^b m\,\vec{r}''(t) \cdot \vec{r}'(t)\,dt = m \int_a^b \frac{d}{dt}\left(\frac{\vec{r}'(t) \cdot \vec{r}'(t)}{2}\right) dt \\ &= \frac{m}{2} \int_a^b \frac{d}{dt}\left(\|\vec{r}'(t)\|^2\right) dt = \frac{m}{2}\left[\|\vec{r}'(t)\|^2\right]_a^b\end{aligned}$$

Therefore, **Work** $= \dfrac{m}{2}\left(\|\vec{v}(b)\|^2 - \|\vec{v}(a)\|^2\right) = K(B) - K(A)$

where $K(Q) =$ kinetic energy of the object when it is at point Q.

Conservative Vector Fields in Physics (Optional)

$$\text{Work} = \int_{\mathcal{C}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} = K(B) - K(A)$$

where $K(Q)$ = kinetic energy of the object at point Q.
Let $\vec{\mathsf{F}}$ be a conservative force field with scalar potential $f(x, y, z)$.
The **potential energy** of an object at (x, y, z) is
$P(x, y, z) = -f(x, y, z)$.
By the Fundamental Theorem for Conservative Vector Fields,

$$\int_{\mathcal{C}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} = -\int_{\mathcal{C}} \nabla P \cdot d\vec{\mathsf{r}} = P(A) - P(B)$$

Law of Conservation of Energy

$P(A) + K(A) = P(B) + K(B)$